ANALYSIS 2017.

1. Team

Problem 1.1. Let μ and ν be two finite Borel measures on $\mathbb{R}^n$ such that $\forall x \in \mathbb{R}^n$ and $\forall R > 1$, we have

$$\mu(B(x,R)) = \nu(B(x,R))$$

where $B(x,R) = \{y \in \mathbb{R}^n : |x-y| < R\}$.

(1) Prove that if μ and ν are absolutely continuous with respect to the Lebesgue measure, then $\mu = \nu$.
(2) What if μ and ν are not absolutely continuous with respect to the Lebesgue measure?

Problem 1.2. Let Ω be a bounded simply connected domain. If $f : \Omega \to \Omega$ is analytic and there exists a point $z_0 \in \Omega$ such that

$$f(z_0) = z_0, \quad \text{and} \quad f'(z_0) = 1$$

then $f(z) = z$ for all $z \in \Omega$.

Problem 1.3. Find all the solutions of

$$\begin{cases} \Delta u = 0, \quad u > 0 \quad \text{in} \quad \{(x,y) : x > 0, y > 0\}, \\ u(x,0) = 0, \; u(0,y) = 0, \quad \text{for} \quad x > 0, y > 0. \end{cases}$$

Here we assume all the conditions hold in the classical sense.

Problem 1.4. For $0 \le t \le T (T > 0)$, Ω_t is an open bounded region in $\mathbb{R}^3$ with smooth boundary $\partial\Omega_t$. Set

$$D_\Gamma = \{(x,t) : x \in \Omega_t, 0 \le t \le T\}, \qquad \partial D_t = \{(x,t) : x \in \partial\Omega_t, 0 \le t \le T\}.$$

Let $\eta(x,t) = (\eta_0, \eta_1, \eta_2, \eta_3)(x,t)$ be the unit outward normal of ∂D_Γ at (x,t). Suppose that $v(x,t) = (v_1, v_2, v_3)(x,t)$ is a smooth vector field and $P(x,t)$ is a smooth scalar function defined on D_Γ satisfying

$$\begin{cases} \frac{\partial v_i}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial P}{\partial x_i} = 0, \quad i = 1,2,3 \quad \text{in} \quad D_\Gamma, \\ \sum_{j=1}^3 \frac{\partial v_j}{\partial x_j} = 0, \quad \text{in} \quad D_\Gamma, \end{cases}$$

$$\begin{cases} \eta_0 + \sum_{j=1}^3 v_j \eta_j = 0, \qquad \text{on} \quad \partial D_\Gamma \\ P = 0, \qquad \text{on} \quad \partial D_\Gamma. \end{cases}$$

(1) Prove that $\int_{\Omega_t} |v|^2(x,t)dx$ is a constant for $0 < t < T$.
(2) Suppose that

$$\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = 0$$

for all $1 \le i,j \le 3$ in D_Γ, and

$$\frac{\partial v_{i_0}(x,t_0)}{\partial x_{j_0}} \neq 0$$

for $x \in \Omega_{t_0}$ and $t_0 \in (0, T)$ and some $i_0, j_0 \in \{1, 2, 3\}$. Prove that

$$\frac{\partial P}{\partial n} < 0$$

on $\partial\Omega_{t_0}$, where $n = (n_1.n_2, n_3)$ is the unit outer normal to $\partial\Omega_{t_0}$.